

A correction of a characterization of planar partial cubes

Rémi Desgranges* Kolja Knauer†

October 19, 2016

Abstract

In this note we determine the set of expansions such that a partial cube is planar if and only if it arises by a sequence of such expansions from a single vertex. This corrects a result of Peterin.

1 Introduction

A graph is a *partial cube* if it is isomorphic to an isometric subgraph G of a hypercube graph Q_d , i.e., $\text{dist}_G(v, w) = \text{dist}_{Q_d}(v, w)$ for all $v, w \in G$. Any isometric embedding of a partial cube into a hypercube leads to a the same partition of edges into so-called θ -classes, where two edges are equivalent, if they correspond to a change in the same coordinate of the hypercube. This can be shown using the Djoković-Winkler-relation θ which is defined in the graph without reference to an embedding, see [4, 5].

Let G^1 and G^2 be two isometric subgraphs of a graph G that (edge-)cover G and such that their intersection $G' := G^1 \cap G^2$ is non-empty. The *expansion* H of G with respect to G^1 and G^2 is obtained by considering G^1 and G^2 as two disjoint graphs and connecting them by a matching between corresponding vertices in the two resulting copies of G' . A result of Chepoi [2] says that a graph is a partial cube if and only if it can be obtained from a single vertex by a sequence of expansions. An equivalence class of edges with respect to θ in a partial cube is an inclusion minimal edge cut. The inverse operation of an expansion in partial cubes, consist in taking a θ -class of edges E_f and contracting it. The two disjoint copies of the corresponding G^1 and G^2 are just the two components of the graph where E_f is deleted.

2 The flaw and the result

Let H be an expansion of a planar graph G with respect to G^1 and G^2 . Then H is a *2-face expansion* of G if G^1 and G^2 have plane embeddings such that $G' := G^1 \cap G^2$ lies on the outer face, respectively. Peterin [3] proposes a theorem stating that a graph is a planar partial cube if and only if it can be obtained

*ENS Cachan, Université Paris-Saclay

†Laboratoire d'Informatique Fondamentale, Aix-Marseille Université and CNRS, Faculté des Sciences de Luminy, F-13288 Marseille Cedex 9, France

from a single vertex by a sequence of 2-face expansions. However, his argument has a flaw. Indeed, not all 2-face expansions preserve planarity as shown by the example in Figure 1.

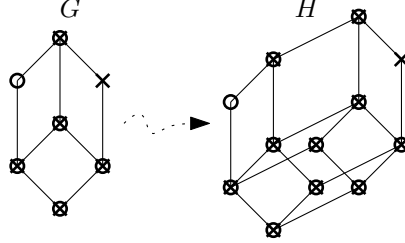


Figure 1: A non-planar 2-face expansion H of a planar partial cube G , where G^1 and G^2 are drawn as crosses and circles, respectively.

The correct concept are non-crossing 2-face expansions: We call a 2-face expansion *non-crossing* if the plane embeddings of G^1 and G^2 furthermore have the property that the orderings on G' obtained from traversing the outer faces of G^1 and G^2 in the clockwise order, respectively, are opposite.

Theorem 1. *A graph is a planar partial cube if and only if it can be obtained from a single vertex by a sequence of non-crossing 2-face expansions.*

Proof. Let G be a planar partial cube and G^1 and G^2 two subgraphs satisfying the preconditions for doing a non-crossing 2-face expansion. We can thus embed G^1 and G^2 disjointly into the plane such that the two copies of $G' := G^1 \cap G^2$ appear in opposite order around their outer face, respectively. Connecting corresponding vertices of the two copies of G' by a matching does not create crossings, because the 2-face expansion is non-crossing.

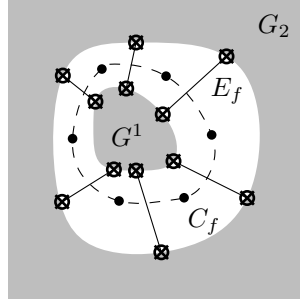


Figure 2: Two disjoint copies of subgraphs G^1 and G^2 in a planar partial cube H .

Let conversely H be a partial cube with a plane drawing and E_f a θ -class of edges in H . Since H is a partial cube, E_f is an inclusion-minimal edge cut of H . Thus, $H \setminus E_f$ has precisely two components, which when identified along the end vertices of edges in E_f give G^1 and G^2 in $G = H/E_f$. Since E_f is a minimal cut its planar dual is a simple cycle C_f , having G^1 and G^2 in its interior and exterior, respectively. In particular, the copies of G' in G^1 and G^2 , respectively, can be connected with an edge to C_f , without crossing any other feature of G^1

and G^2 , respectively. Thus, the copies of G' lie on the outer face of G^1 and on an inner face of G^2 , respectively. Moreover, following E_f in the sense of clockwise traversal of C_f gives an order on the two copies of G' on the outer face of G^1 and an inner face F of G^2 . See Figure 2 for an illustration. Changing the drawing of G^2 such that F becomes the outer face, gives the required embedding of disjoint copies of G^1 and G^2 . Since the choice of E_f was arbitrary, we have shown that all expansions leading to H must be non-crossing 2-face expansions. \square

3 Remarks

We have characterized planar partial cubes graphs by expansions. Planar partial cubes have also been characterized in a topological way as dual graphs of non-separating pseudodisc arrangements [1]. There is a third interesting way of characterizing them. The class of planar partial cubes is closed under *partial cube minors*, i.e., contraction of G to G/E_f where E_f is a θ -class and restriction to a component of $G \setminus E_f$. What is the family of minimal obstructions for a partial cube to being planar, with respect to this notion of minor?

Acknowledgements:

We wish to thank Iztok Peterin for discussing this result with us.

References

- [1] M. ALBENQUE AND K. KNAUER, *Convexity in partial cubes: the hull number.*, Discrete Math., 339 (2016), pp. 866–876.
- [2] V. CHEPOJ, *Isometric subgraphs of Hamming graphs and d -convexity.*, Cybernetics, 24 (1988), pp. 6–11.
- [3] I. PETERIN, *A characterization of planar partial cubes.*, Discrete Math., 308 (2008), pp. 6596–6600.
- [4] D. Ž. DJOKOVIĆ, *Distance-preserving subgraphs of hypercubes.*, J. Comb. Theory, Ser. B, 14 (1973), pp. 263–267.
- [5] P. M. WINKLER, *Isometric embedding in products of complete graphs.*, Discrete Appl. Math., 7 (1984), pp. 221–225.